Kuwait University	Math 101	Date:	June 5, 2008
Dept. of Math. & Comp. Sci.	Final Exam	Duration:	Two Hours

Calculators, mobile phones, pagers and all other mobile communication equipments are not allowed

Answer the following questions. Each question weighs 4 points.

1. Evaluate the following limits, if they exist:

   (a) $\lim\limits_{x\to 0} \dfrac{1}{x}\left(\dfrac{1}{x+5} - \dfrac{1}{5}\right)$ (b) $\lim\limits_{x\to 0} \dfrac{\sin 6x}{x \cos 3x}$

2. Find equations of the lines of slope -4 that are tangents to the curve $y = \dfrac{1}{x}$.

3. Let $f(x) = \dfrac{x^2 + x}{|x|}$, where $x \neq 0$. Can f be defined at $x = 0$ so that, f becomes continuous? Justify your answer.

4. A farmer has 200 meters of fence to be used in constructing three sides of a rectangle. An existing long straight wall is to be used for the fourth side. What dimensions will maximize the area of the rectangle?

5. Show that the equation $2x^5 + x - 1 = 0$ has exactly one real solution.

6. Find the local extrema of $f(x) = \displaystyle\int_0^{x^2} \dfrac{1}{\sin t + 5}\, dt$, $x \in \mathbb{R}$.

7. Find the average value, f_{av}, of $f(x) = |x|$ on the interval $[-1, 2]$. Find the point $a \in [-1, 2]$, that satisfies the conclusion of the Mean Value Theorem for Integrals.

8. Evaluate: (a) $\displaystyle\int \dfrac{x^3 - 1}{(x^4 - 4x)^{\frac{2}{3}}}\, dx$ (b) $\displaystyle\int_0^{\frac{\pi}{2}} (1 + \sin t)^3 \cos t\, dt$

9. Find the area of the region bounded by the curves $x = y^2$ and $x = y^3$.

10. The region bounded by the curves $y = 4 - x^2$ and $y = 0$ is revolved about:

    (a) the line $x = 3$,

    (b) the line $y = -2$.

    Set up an integral that can be used to find the volume of the resulting solid in each case.

1. (a) $\lim\limits_{x\to 0}\dfrac{1}{x}\left(\dfrac{1}{x+5}-\dfrac{1}{5}\right)=\lim\limits_{x\to 0}\dfrac{-x}{5x\,(x+5)}=\lim\limits_{x\to 0}\dfrac{-1}{5\,(x+5)}=\boxed{-\frac{1}{25}}$.

(b) Solution 1: $\lim\limits_{x\to 0}\dfrac{\sin 6x}{x\cos 3x}=\lim\limits_{x\to 0}\dfrac{2\sin 3x\cos 3x}{x\cos 3x}=2\times 3\lim\limits_{x\to 0}\dfrac{\sin 3x}{3x}=\boxed{6}$ $(\lim\limits_{x\to 0}3x=0)$

Solution 2: $\lim\limits_{x\to 0}\dfrac{\sin 6x}{x\cos 3x}=\lim\limits_{x\to 0}\dfrac{6\dfrac{\sin 6x}{6x}}{\cos 3x}=\dfrac{6\lim\limits_{x\to 0}\dfrac{\sin 6x}{6x}}{\lim\limits_{x\to 0}\cos 3x}=\boxed{6}$

$(\lim\limits_{x\to 0}\cos 3x=1$ & $\lim\limits_{x\to 0}6x=0)$

2. $y'=-\dfrac{1}{x^2}, y'=-4$ at $x=\pm\frac{1}{2}\implies y'=-4$ at $\left(-\frac{1}{2},-2\right)$ or $\left(\frac{1}{2},2\right)$.

Equations of tangent line are: $\boxed{y+2=-4\left(x+\frac{1}{2}\right)\ \&\ y-2=-4\left(x-\frac{1}{2}\right)}$

3. $\lim\limits_{x\to 0^{\pm}}f(x)=\lim\limits_{x\to 0^{\pm}}\dfrac{x\,(x+1)}{\pm x}=\boxed{\pm 1}$. $\implies f$ has a *jump* discontinuity at $x=0$ but not a removable discontinuity. So, It cannot be continuous for any value of f at $x=0$.

4. $2x+y=200$, Area $=A=xy=x\,(200-2x)=200x-2x^2\implies\dfrac{dA}{dx}=200-4x$.
$x=50$ is the only critical number of A, $\left.\dfrac{d^2A}{dx^2}\right|_{x=50}=-4<0\implies A$ is maximum at $x=50$, $y=100$ and $A_{\max}=5000\text{ m}^2$.

5. Let $f(x)=2x^5+x-1$..

<u>(I) The exsistence of the root:</u> f is contionuous on $[0,1]$, $f(0)=-1<0$ & $f(1)=2>0$. Form the Intermadiate Value Theorem $\exists\ a\in(0,1)$ such that $f(a)=0$, i.e., the equation has a real solution in $(0,1)$.

[OR, f is continuous on $\mathbb{R}$, $\lim\limits_{x\to\pm\infty}f(x)=\pm\infty$, then f has a real root]

<u>(II) The uniqueness of the root:</u> $f'(x)=10x^4+1>0$ for all $x\in\mathbb{R}$. So, f is increasing and since f has a root in $(0,1)$, then this root must be unique.

OR By using Rolle's Theorem. Suppose contrarily, x_1,x_2 are two distinct roots ($x_1<x_2$, say). Thus, $f(x_1)=0=f(x_2)$, f is continuous on $[x_1,x_2]$ and differentiable on (x_1,x_2) (f is polynomial). Therefore, from Rolle's Theorem, $\exists\ c\in(x_1,x_2)$, such that $f'(c)=0$, but, $10c^4+1>0$, which contradicts Rolle's Theorem.

6. $f'(x)=\dfrac{d}{dx}\left[\displaystyle\int_0^{x^2}\dfrac{1}{\sin t+5}dt\right]=\boxed{\frac{2x}{\sin x^2+5}}$. Since, $\sin x^2+5\geq 4$, so, the only critical number of f is $x=0$, where $f'(x)=0$.

I	$(-\infty,0)$	$(0,\infty)$
sign of $f'(x)$	$-$	$+$
Conclusion	$\searrow$	$\nearrow$

$f(0)=\displaystyle\int_0^{0^2}\dfrac{1}{\sin t+5}dt=0$, is the local minimum of f.

7. $\displaystyle\int_{-1}^{2}|x|\,dx=\int_{-1}^{0}-x\,dx+\int_0^2 x\,dx=\left.\frac{-x^2}{2}\right]_{-1}^{0}+\left.\frac{x^2}{2}\right]_0^2=\boxed{\frac{5}{2}}\implies f_{av}=\frac{1}{2-(-1)}\int_{-1}^{2}|x|\,dx=\boxed{\frac{5}{6}}$.

$f(a) = |a| = \frac{5}{6} \implies \boxed{a = \pm\frac{5}{6} \in [-1, 2]}.$

8. (a) Put $u = x^4 - 4x \implies udu = 4(x^3 - 1)\,dx \implies \int \frac{x^3 - 1}{(x^4 - 4x)^{\frac{2}{3}}}\,dx = \frac{1}{4}\int \frac{du}{u^{\frac{2}{3}}} =$
$\frac{3}{4}u^{\frac{1}{3}} + C = \boxed{\frac{3}{4}(x^4 - 4x)^{\frac{1}{3}} + C}$

(b) Put $u = 1 + \sin t, u(0) = 1 \& u\left(\frac{\pi}{2}\right) = 2 \implies udu = \cos t\,dt \implies \int_0^{\frac{\pi}{2}} (1 + \sin t)^3 \cos t\,dt =$
$\int_1^2 u^3\,du = \left.\frac{u^4}{4}\right]_1^2 = \boxed{\frac{15}{4}}$

9. The Area $= \int_0^1 (y^2 - y^3)\,dy = \left.\frac{y^3}{3} - \frac{y^4}{4}\right]_0^1 = \frac{1}{3} - \frac{1}{4} = \boxed{\frac{1}{12}}$

10. (a) Revolution about the line $x = 3$:

$$Volume = 2\pi\int_{-2}^{2} (3 - x)(4 - x^2)\,dx$$

OR $$Volume = \pi\int_0^4 \left[\left(3 + \sqrt{4 - y}\right)^2 - \left(3 - \sqrt{4 - y}\right)^2\right]\,dy.$$

(b) Revolution about the line $y = -2$:

$$Volume = \pi\int_{-2}^{2} \left[(6 - x^2)^2 - (2)^2\right]\,dx$$

OR $$Volume = 2\pi\int_0^4 (y + 2)\left(2\sqrt{4 - y}\right)\,dy.$$